Chapter 7

Inversive Geometry

One of the most stunning products of the revival of Euclidean geometry in the 19th century is the method of inversion, introduced by L.J. Magnus in 1831. The power of inversion lies in its ability to convert statements about circles into statements about lines, often reducing the difficult to the trivial.

7.1 Inversion

Let O be a point in the plane and r a positive real number. The *inversion* with center O and radius r is a transformation mapping every point $P \neq O$ in the plane to the point P' on the ray $\vec{OP}$ such that $OP \cdot OP' = r^2$. Since specifying a point and a positive real number is the same as specifying a circle (the point and the positive real corresponding to the center and radius, respectively, of the circle), we can also speak of inversion through a circle using the same definition. DIAGRAM

What happens to the point O? Points near O get sent very far away, in all different directions, so there is no good place to put O itself. To rectify this, we define the *inversive plane* as the usual plane with one additional point, called the *point at infinity.* (We will use the label ∞ throughout this chapter for the point at infinity.) We extend inversion to the entire inversion plane by declaring that O and ∞ are inverses of each other.

As an aside, we note a natural interpretation of the inversive plane. Under stereographic projection (used in some maps), the surface of a sphere, minus the North Pole, is mapped to a plane tangent to the sphere at the South Pole as follows: a point on the sphere maps to the point on the plane collinear with the given point and the North Pole. Then the point at infinity corresponds to the North Pole, and the inversive plane corresponds to the whole sphere. In fact, inversion through the South Pole with the appropriate radius corresponds to reflecting the sphere through the plane of the equator! DIAGRAM.

Returning to Euclidean geometry, we now establish some important properties of inversion. We first make an easy but important observation.

Fact 7.1. *If O is the center of an inversion taking P to P' and Q to Q', then the triangles OPQ and $OQ'P'$ are oppositely similar.*

In particular, we have that $\angle OP'Q' = -\angle OQP$, a fact underlying our next proof.

Theorem 7.2. *The image of a (generalized) circle under an inversion is a (generalized) circle.*

Proof. Let A, B, C, D be four concyclic points and A', B', C', D' their images under some inversion about O. We now chase directed angles, using the similar triangles of Fact 7.1:

$$\begin{aligned} \angle A'B'C' &= \angle A'B'O + \angle OB'C' \\ &= \angle BAO + \angle OCB \\ &= \angle BAD + \angle DAO + \angle OCD + \angle DCB \\ &= \angle DAO + \angle OCD \\ &= \angle A'D'O + \angle OD'C' \\ &= \angle A'D'C'. \end{aligned}$$

We see that A', B', C', D' are concyclic as well. DIAGRAM. □

Notice the way the angles are broken up and recombined in the above proof. In some cases, inversion can turn a constraint involving two or more angles in different places into a constraint about a single angle, which then is easier to work with. Some examples can be found in the problems.

Inversion also turns out to "reverse the angles between lines". Since lines are sent to circles in general, we will have to define the angle between two circles to make sense of this statement.

Given two circles ω_1 and ω_2, the (directed) angle between them at one of their intersections P is defined as the (directed) angle from the tangent to ω_1 at P to the tangent of ω_2 at P. In particular, two circles are *orthogonal* if the angle between them is a right angle. Note that the angle between the two circles is only well-defined up to sign without a choice of a point of intersection, since the angle at the other intersection is reversed. (Of course, orthogonality does not depend on this choice.) Note also that a line and a circle are orthogonal if and only if the line passes through the center of the circle.

Fact 7.3. *The directed angle between circles (at a chosen intersection) is reversed under inversion.*

Distances don't fare as well under inversion, but one can say something using Fact 7.1.

Fact 7.4 (Inversive distance formula). *If O is the center of an inversion of radius r sending P to P' and Q to Q', then*

$$P'Q' = PQ \cdot \frac{r^2}{OP \cdot OQ}.$$

Problems for Section 7.1

1. Deduce Theorem 7.2 from Problem 1.3.3 (or use the above proof to figure out how to do that exercise).

2. Give another proof of Theorem 7.2 using the converse of the power-of-a-point theorem (Fact 4.2) and Fact 7.4.

3. The angle between two lines through the origin is clearly preserved under inversion. Why doesn't this contradict the fact that inversion reverses angles?

4. (IMO 1996/2) Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

   Let D, E be the incenters of triangles APB, APC, respectively. Prove that AP, BD, CE meet in a point. (Many other solutions are possible; over 25 were submitted by contestants at the IMO!)

5. (IMO 1998 proposal) Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360°$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

   Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

6. Prove that the following are equivalent:

   1. The points A and B are inverses through the circle ω.
   2. The line AB and the circle with diameter AB are both orthogonal to ω.
   3. ω is a circle of Apollonius with respect to A and B.

   In particular, conclude that a circle distinct from ω is fixed (as a whole, not pointwise) by inversion through ω if and only if it is orthogonal to ω.

7. Show that a set of circles is coaxal if and only if there is a circle orthogonal to all of them. Deduce that coaxal circles remain that way under inversion. Also, try drawing a family of coaxal circles and some circles orthogonal to them; the picture is very pretty.

8. Prove that any two nonintersecting circles can be inverted into concentric circles.

7.2 The power of inversion

Steiner was able to give nearly trivial proofs of some very hard-looking statements using inversion. In this section, we take a quick look at some of his dazzling results.

We start with a classical result attributed to Pappus. The figure formed by the three semicircles is known as an *arbelos* ("shoemaker's knife" in Greek), and was a favorite of Archimedes. (It was also a favorite of one-time USA IMO coach Samuel Greitzer, who for several years authored a journal for high school students of the same name.)

Theorem 7.5 (Pappus). *Let ω be a semicircle with diameter AB. Let ω_1 and ω_2 be two semicircles externally tangent to each other at C, and internally tangent to ω at A and B, respectively. Let $C_1, C_2, \dots$ be a sequence of circles, each tangent to ω and ω_1, such that C_i is tangent to C_{i+1} and C_1 is tangent to ω_2, as in the diagram. Let r_n be the radius of C_n and d_n the distance from the center of C_n to AB. Then for all n,*

$$d_n = 2nr_n.$$

DIAGRAM

Proof. Perform an inversion with center A, and choose the radius of inversion so that C_n remains fixed. Then ω and ω_1 map to lines perpendicular to AB and tangent to C_n, and $C_{n-1}, \dots, C_1$ to a column of circles between the lines, with ω_2' at the bottom of the column. The relation $d_n = 2nr_n$ is now obvious. □

The following theorem is known as *Steiner's porism.*

Theorem 7.6. *Suppose two nonintersecting circles have the property that one can fit a "ring" of n circles between them, each tangent to the next, as in the diagram. DIAGRAM. Then one can do this starting with any circle tangent to both given circles.*

Proof. By Problem 7.1.8, a suitable inversion takes the given circles to concentric circles, while preserving tangency of circles. The result is now obvious. □

Problems for Section 7.2

1. Suppose that, in the hypotheses of Pappus' theorem, we assume that C_0 is tangent to ω, ω_1 and the line AB (instead of the semicircle ω_2). Show that in this case $d_n = (2n-1)r_n$.

2. (Romania, 1997) Let ω be a circle and AB a line not intersecting ω. Given a point P_0 on ω, define the sequence $P_0, P_1, \dots$ as follows: P_{n+1} is the second intersection with ω of the line through B and the second intersection of the line AP_n with ω. Prove that for a positive integer k, if $P_0 = P_k$ for some choice of P_0, then $P_0 = P_k$ for any choice of P_0.

7.3 Inversion in practice

So much for the power of inversion; how is it useful for real problems? The remainder of this chapter will be devoted to several examples of how inversion can be used to solve olympiad-style problems. The paradigm will almost always be: invert the given information, invert the conclusion, and proceed to solve the new problem. Beware that in somes, though, it may be necessary to superimpose the original and inverted diagrams (as in the proof of Theorem 7.5), or to compare the original and inverted diagrams (e.g. using Fact 7.4).

A general principle behind this method is that problems with few circles are easier than those with many circles. Hence when inverting, one should find a "busy point," one with many circles and lines going through it, and invert there.

Problems for Section 7.3

1. Make up an inversion problem by reversing the paradigm: start with a result that you know, invert about some point, and see what you get. The tricky part is choosing things well enough so that the resulting problem doesn't have an obvious busy point; such a problem would be too easy!
2. Given circles C_1, C_2, C_3, C_4 such that C_i and C_{i+1} are externally tangent for $i = 1, 2, 3, 4$ (where $C_5 = C_1$). Prove that the four points of tangency are concyclic.
3. (Romania, 1997) Let ABC be a triangle, D a point on side BC and ω the circumcircle of ABC. Show that the circles tangent to ω, AD, BD and to ω, AD, DC are tangent to each other if and only if $\angle BAD = \angle CAD$.
4. (Russia, 1995) Given a semicircle with diameter AB and center O and a line which intersects the semicircle at C and D and line AB at M ($MB < MA$, $MD < MC$). Let K be the second point of intersection of the circumcircles of triangles AOC and DOB. Prove that angle MKO is a right angle.
5. (USAMO 1993/2) Let $ABCD$ be a convex quadrilateral with perpendicular diagonals meeting at O. Prove that the reflections of O across AB, BC, CD, DA are concyclic. (For an added challenge, find a non-inversive proof as well.)
6. (Apollonius' problem) Given three nonintersecting circles, how many circles are tangent to all three? And how can they be constructed with straightedge and compass?
7. (IMO 1994 proposal) The incircle of ABC touches BC, CA, AB at D, E, F, respectively. X is a point inside ABC such that the incircle of XBC touches BC at D also, and touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

8. (Israel, 1995) Let PQ be the diameter of semicircle H. Circle O is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that AB is perpendicular to PQ and is also tangent to O. Prove that AC bisects $\angle PAB$.

9. (Ptolemy's inequality) If $ABCD$ is a convex quadrilateral, then

$$AC \cdot BD \leq AB \cdot CD + BC \cdot DA,$$

with equality if and only if $ABCD$ is cyclic. (See also Theorem A.11.)

10. (IMO 1993/2) Let A, B, C, D be four points in the plane, with C, D on the same side of line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = \pi/2 + \angle ACB$. Find the ratio $(AB \cdot CD)/(AC \cdot BD)$ and prove that the circumcircles of triangles ACD and BCD are orthogonal.

11. (Iran, 1995) Let M, N, P be the points of intersection of the incircle of $\triangle ABC$ with sides BC, CA, AB, respectively. prove that the orthocenter of $\triangle MNP$, the incenter of $\triangle ABC$, and the circumcenter of $\triangle ABC$ are collinear. (The paradigm does not hold here: invert through the incircle, then superimpose the original and inverted diagrams.)

12. (MOP 1997) Let ABC be a triangle and O its circumcenter. The lines AB and AC meet the circumcircle of triangle BOC again at B_1 and C_1, respectively. Let D be the intersection of lines BC and B_1C_1. Show that the circle tangent to AD at A and having its center on B_1C_1 is orthogonal to the circle with diameter OD.